# The Arithmetic to Algebra Gap

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## What Math Knowledge is Needed to Solve these Equations?

$2x + 5 = 18$

$\quad - 5 \quad - 5$

$2x = 13$

$\frac{2x}{2} = \frac{13}{2}$

$1x = 6\,1/2$

$(y - 5)(y + 2)$

$(y)(y) + (y)(2) - (5)(y) - (5)(2)$

$y^2 + 2y - 5y - 10$

$y^2 - 3y - 10$



## Can we overcome a weak start in mathematics?

- Morgan, Farkas, and Wu (2009) found that 70% of the students in the lowest 10% in kindergarten remain that way five years later.
  - Translation= interventions only helped 30% of the lowest performing students identified early.
- "Students who fail to acquire the pieces that make up number sense as early as kindergarten are at the greatest risk for failure in mathematics in the long term" (Witzel, Riccomini, & Herlong, 2013, p. 5).

## Math Proficiency of U.S. Students



- International comparisons
- Low fractions of proficiency on NAEP
- Falling proficiency at higher grades
  - Heavy remedial demand upon entry into college
  - Achievement gap

***Algebra as a gateway***



## Mathematics Performance

**Translated to Real World Performance**

78% of adults cannot explain how to compute interest paid on a loan

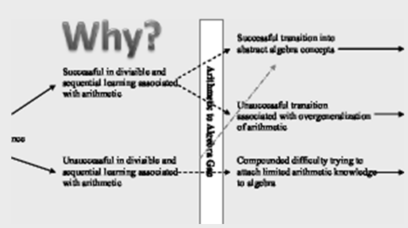
- 71% cannot calculate miles per gallon
- 58% cannot calculate a 10% tip

Mathematics Advisory Panel Final Report, 2008

## Arithmetic to Algebra Gap (SH#1)

(Witzel, Smith, & Brownell, 2001)



Not only do parents and students feel like the bees in the cartoon, but so do many teachers, both general and special education.



## Topics for Today

Overarching goal: Designing instruction to help students of all skill levels achieve success in mathematics.

- NMAP, 2008 Final Report
- Components of Effective Math Instruction
  - Concrete to Representational to Abstract Sequence of Instruction
  - Math Progressions



## How one learns affects how one teaches




## Characteristics of those who struggle the most

- Exhibit problems with memory (working memory)
- In attentive and unable to focus on a task
- Lack of strategic approaches to mathematics
  - Disorganized, impulsive, unaware of where to begin an assignment
  - Unaware of possible steps to break the problem into a manageable task, possibly due to the magnitude of the task
  - Experience feelings of frustration, failure, or anxiety
  - Lack persistence to solve longer problems or personal struggles
  - Attribute failure to uncontrollable factors (e.g., luck, teacher's instructional style)



## Learning Processes-NMAP-2008

- To prepare students for Algebra, the curriculum must ***simultaneously*** develop ***conceptual understanding, computational fluency factual knowledge*** and ***problem solving skills***
- Limitations in the ability to keep many things in mind (***working-memory***) can hinder mathematics performance.
  - \- ***Practice*** can offset this through automatic recall, which results in less information to keep in mind and frees attention for new aspects of material at hand.
  - \- Learning is most effective when ***practice is combined with instruction*** on related concepts.
  - \- Conceptual understanding ***promotes transfer*** of learning to new problems and better long-term retention.

NMAP, 2008



## Instructional Practices-NMAP-2008

Research on students who are ***low achievers, have difficulties in mathematics, or have learning disabilities*** related to mathematics tells us that the effective practice includes:

- ***Systematic instruction available on a regular basis***
- ***Clear problem solving models***
- ***Carefully orchestrated examples/ sequences of examples.***
- ***Concrete representations to prepare abstract approach and notation.***
- ***Participatory thinking aloud by students and teachers.***

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## 16 Elements of Explicit Instruction

(Archer & Hughes, 2011)

Elements 1-8

1. Focus instruction on critical content
2. Sequence skills logically
3. Task analyze complex skills into smaller steps
4. Design focused lessons
5. Set the expectation to start the lesson
6. Review prior skills
7. Demonstrate stepwise instructions
8. Use clear and concise language



## 16 Elements of Explicit Instruction

(Archer & Hughes, 2011)

9. Provide examples and nonexamples
1. Provide students guided practice
2. Require frequent responses
3. Monitor student performance closely
4. Provide immediate feedback (corrective or affirmation)
5. Deliver instruction at a brisk pace
6. Connect information across lessons and content
7. Provide abundant time for practice and cumulative review



## Avoid Tricks: converting fractions

Convert this mixed fraction into an improper fraction. 16

4 $^2/_5$

How did you know how to do it?

Did you…

a. 4x5

b. + 2 = 22

c $^{22}/_5$ Why ?

**Say, "Four and two – fifths"**

$^4/_1 + ^2/_5$ or $^{20}/_5 + ^2/_5 = ^{22}/_5$.



## Avoid Tricks:
## Division of Fractions

- Why is it that when you divide fractions, the answer is larger? Also, why do you invert and multiply?

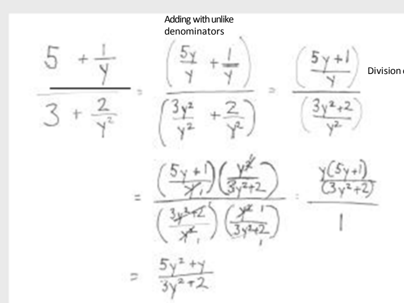
- $^2/_3$ divided by $^1/_4 = ^2/_3(^4/_1) = ^8/_3$

$^2/_3(^4/_1)^8/_3{}^8/_3$

$^1/_4(^4/_1)4/4\ 1/1$



## Why teach the basics correctly

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## Make sense of complex math:
## $z^3 - z^2 - 9z + 9$

- $z^2(z-1) - 9(z-1)$
- $(z^2-9)(z-1)$
  - And $z^2-9$ is a difference of sq.
  - $(z-3)(z+3)(z-1)$

Set z = 5

- $5^2(5-1) - 9(5-1)$
- $(5^2-9)(5-1)$
- And $5^2-9$ is a difference of squares

check

$(5-3)(5+3)(5-1) = 5^3 - 5^2 - 9(5) + 9$

$(2)\ (8)\ (5-1) = 125 - 25 - 45 + 9$

$64 = 64$



## Standards Based Classroom

**Daily Lesson Structure –**

**ALL GRADE LEVELS**

1. Fluency Activity/Homework Review/Activator
2. Review of Standards and Essential Vocabulary
3. **Work Sessions-Flexible Grouping &**
   - **Small Group Teacher Directed Instruction**
   - **Station Teaching**
4. Closure & Summary

Me, We, Two, You

 

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## CRA as effective instruction

(Gersten et al, 2009; NMP, 2008; Riccomini & Witzel, 2010; Witzel, 2005)

Concrete to Representational to Abstract Sequence of Instruction (CRA)

- Concrete (expeditious use of manipulatives)
- Representations (pictorial)
- Abstract procedures

Excellent for teaching accuracy and understanding

Example:
http://engage.ucf.edu/v/p/2wKBsbBfcit.usf.edu/mathvids/strategies/cra.html



## Manipulative Objects:
## A Number Sense Teaching Tool

- Researched benefits span over:
  - Developing numeration -Basic facts -Fractions
  - negative #s -Area & perimeter -3D figures
  - Manipulative objects do NOT teach children.... Teachers do!
- Some helpful hints
  - Practice using manipulatives before you teach
  - Provide language experiences while using manipulatives
  - Develop a pictorial representation for transition to abstract understanding... the ultimate purpose



## Concrete Misconceptions

- Manipulative objects teach children
  - Any concrete object transfers to abstract understanding
  - All students will achieve higher gains in math scores when taught using concrete objects
  - All math can be taught through the use of concrete objects
  - Math teachers are united in their belief in manipulative instruction

## USDOE on the Use of visual representations

- IES Practice Guide on RtI Math:
  - "Recommendation 5. Interventions should include opportunities for students to work with visual representations of mathematical

    representations of mathematical ideas" (2009).
- IES Practice Guide on Fractions:
  - "Recommendation 2. Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward" (2010)
- IES Practice Guide on Word Problems:
  - "Teach students how to use visual representations" (2012)



## Algebra Intervention Meta-analysis

(Hughes, Fries, Riccomini, & Witzel, in-review)

- Cognitive and model-based problem solving ES= 0.693
- Concrete-Representation-Abstract ES=0.431
- Peer Tutoring ES=0.102
- Graphic Organizers alone ES=0.106
- Technology ES=0.890
- Single-sex instruction ES=0.090

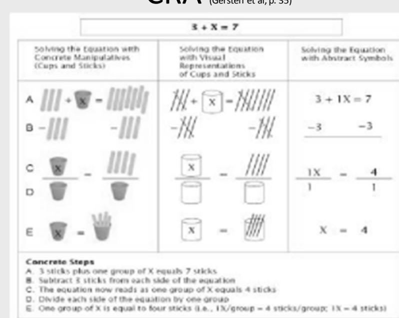


## #5 **Include visual representations of mathematical ideas** (IES Practice Guide, 2009)

- Visual representations used in conjunction with Tier 1 level instruction
- CRA interventions used to support the core instruction and student growth

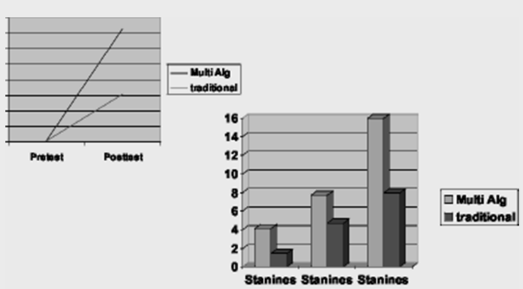


## CRA (Gersten et al, p. 35)



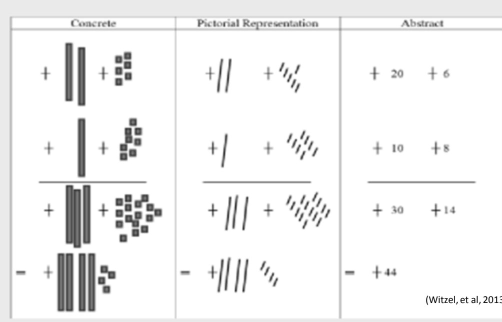



## Multisensory Algebra success





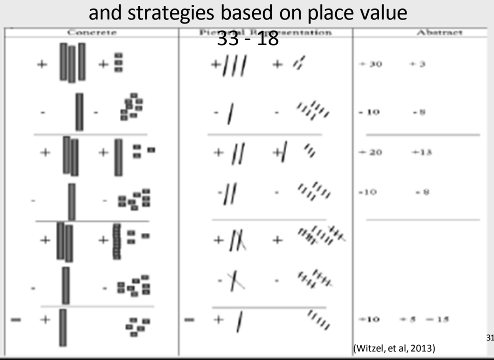

## CCSS 1.NBT.4 Add within 100 using models and strategies based on place value 26 + 18




(Witzel, et al, 2013)

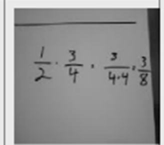
CCSS 2.NBT.7 Add and subtract within 100 using models
and strategies based on place value

33 - 18



(Witzel, et al, 2013)

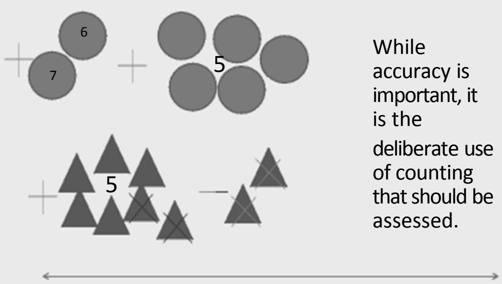


| Concrete | Representational | Abstract |
|---|---|---|
|  |  |  |


Witzel, 2013



## Computation:
## Addition and Subtraction




While accuracy is important, it is the deliberate use of counting that should be assessed.


Witzel, 2013



## Fractions as a Predictor

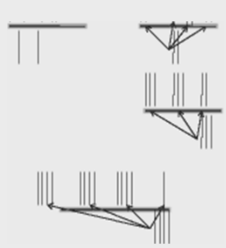
Carnegie Mellon
(http://www.cmu.edu/news/stories/archives/2012/june/june15_mathsuccess.html)

- Siegler et al (2013) found that 5th graders' facility with fractions and division predicted high school students' knowledge of algebra and overall math achievement
- The prediction was even after statistically accounting for parents' education and income and for the children's own age, gender, I.Q., reading comprehension, working memory, and knowledge of whole number addition, subtraction and multiplication.



## Mixed to improper and back: concrete

- 8÷2
- 8 ÷ 3
- 13 ÷ 4





## Intervention with Fractions procedures

(Witzel & Riccomini, 2009)

$^2/_3 + ^1/_2$


(2 + 2) (1 + 1 + 1) + 7
+ =
(3 + 3) (2 + 2 + 2) 6

## Aim interventions at procedural processes (Witzel & Riccomini, 2009)

$^{1}/_{3} - ^{1}/_{3}$




## Fractions Data

Tier 2 Fractions Research

6th Grade Students in 3 different states and 3 interventionists

CRA N=34; Abstract N=37

CRA outscored Abstract only Intervention students in the posttest and 6-week follow-up in adding and subtracting fractions with like and unlike denominators.



## Why would CRA be effective?

(Witzel, Riccomini, & Schneider, 2008)

- Multimodal forms of math acquisition to aid memory and retrieval
- Multiple learning styles are being met to aid relevance and motivation
- Meaningful manipulations of materials allow students to rationalize abstract mathematics
- Procedural accuracy; provides an alternative to algorithm memorization of math rules
- Transportable without concrete materials



## **Topics for Today**

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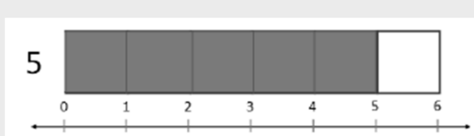
## Number Lines:

## Count to five using blocks (Witzel, 2013)






## Number Line Progressions

Is red greater than or less than green? Is A greater than or less than B?

A

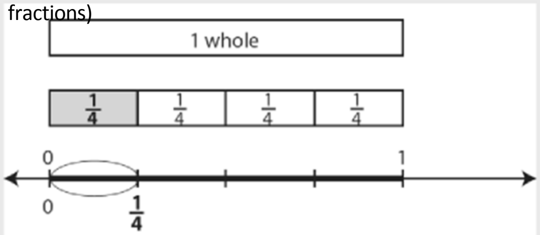
B

(Beckman, 2012)

**Research in the Area of Fractions and Its Application to Classroom Practices**

## Definition of fraction

First define fractions with numerator 1 (unit fractions)






(Beckman, 2012)

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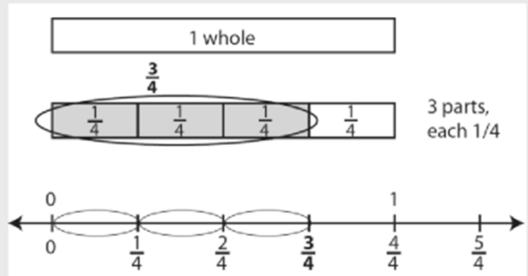
## Definition of fraction

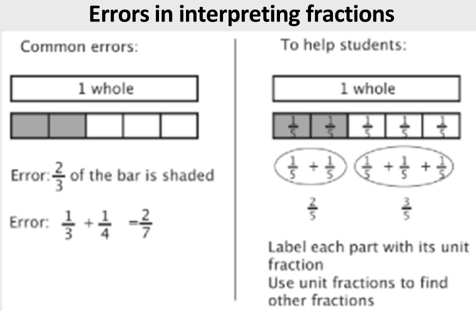





(Beckman, 2012)

**Research in the Area of Fractions and Its Application to Classroom Practices**

## Errors in interpreting fractions

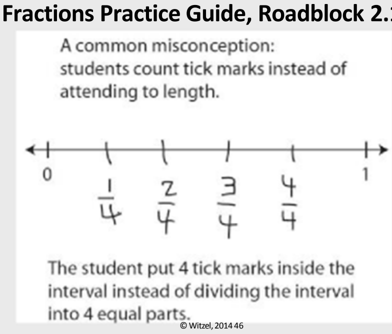





(Beckman, 2012)

**Research in the Area of Fractions and Its Application to Classroom Practices**

## Fractions Practice Guide, Roadblock 2.1





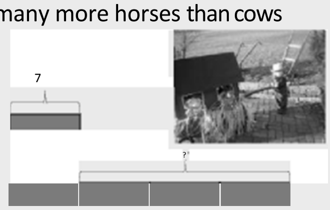

## Implementing Length-based diagrams

(adapted from Beckmann, 2004)

A farmer has 7 cows. She has 4 times as many horses as cows..How many more horses than cows does she have?

Cows

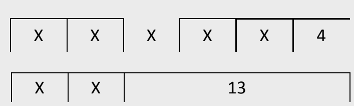
Horses






## Example 1

If 5 times a number increased by 4 , is the same as twice the same number, increased by 13, then what is the number?



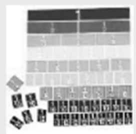

**5x + 4 = 2x + 13**

Witzel & Iiccomini *Arithmetic to Algebra Gap*, due out 2014

## Intervention Part-Whole Model of Fractions

(Fuchs and Schumacher; presented by Gersten, 2012)

- Instruction emphasizes
  – A fraction is part of a whole and always divided into equal parts
  – Definition of numerator and denominator (their role in making 1 number)
- Demonstrated with Fraction Circles and Fraction Tiles



– Shading fractions on various shapes
– Dividing shapes with lines or paper folding
– Dividing a group of objects into fractions



## Findings: Visuals and Graphical Representations

- When teachers used graphical representations to demonstrate problems only, results were much less consistent.
- Visuals were not particularly useful unless students were provided opportunities to *practice* using them.
- Higher effect sizes were for sequential use of representations (e.g., CRA) with clear and explicit stepwise consistency (Gersten et al., 2009; Witzel, Mercer, & Miller, 2003; Witzel, 2005)



## Place Value Progression

- 27 = 2 tens and 7 ones
- 45 = 4 tens and 5 ones

| Hundreds | Tens | Ones |
|---|---|---|
| | | |

- Should be represented physically and verbally
  - Advanced learners should use place value within a calculation exercise.

| Ones | Tenths | Hundredths |
|---|---|---|
| | | |



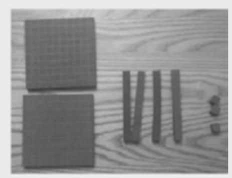
## When you really know place value…

243

How many hundreds are in this number?

How many tens are in this number?

How many ones are in this number?





## Typical Progression Example

2nd Grade: Use place value understanding and properties of operations to perform multi-digit arithmetic. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. (A range of algorithms may be used.)



## Benefits of place value knowledge

13   10 + 3
-7   -7
3 + 3 = 6

341   300 + 40 + 1   200 + 130 + 11
-196   -100 - 90 - 6   -100 - 90 - 6
100 + 40 + 5 = 145

| X | 50 | 3 |
|---|---|---|
| 20 | 1000 | 60 |
| 8 | 400 | 24 |
| 1000+400+60+24= 1484 | | |

## Place Value Progressions:
## Make an array to show 24 x 76

- Fourth grade "Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models."

- Fourth grade "Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models."
- 24x76= ?

| multiply | 70 | 6 |
|---|---|---|
| 20 | 1400 | 120 |
| 4 | 280 | 24 |

1400+120+280+24 = 1824

## Place Value Progressions:
## Make an array to show 2 x 3/4

- Fourth grade "Apply and extend previous understandings of multiplication to multiply a fraction by a whole number."

- Fourth grade "Apply and extend previous understandings of
multiplication to multiply a fraction by a whole number."
- (2)(3/4)

| multiply | 3/4ths |
|---|---|
| 2 | 6/4ths |

(2x3) / 4 = 6/4

## Place Value Progressions:
## Make an array to show $(2\ ^1/_3)(4\ ^1/_2)$

- Fifth grade "Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas."

- Fifth grade "Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas."
- (2 1/3) (4 1/2)

| multiply | 2 | 1/3 |
|---|---|---|
| 4 | 8 | 4/3 |
| 1/2 | 2/2 | 1/6 |

$8 + {}^4/_3 + 1 + {}^1/_6 = 8 + {}^8/_6 + 1 + {}^1/_6 = 10\ {}^3/_6$

## Place Value Progressions:
## Make an array to show 7.6 x 2.4

Fifth grade "Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used."



- Fifth grade "Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used."
- 7.6 x 2.4 = ?

| multiply | 7 | .6 |
|---|---|---|
| 2 | 14 | 1.2 |
| .4 | 2.8 | .24 |

14+1.2+2.8+0.24 = 18.24



## Place Value Progressions:
## Make an array to show (3x - 1)(4x + 5)

Algebra "Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials."



- Algebra "Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials."
- (3x – 1)(4x + 5)

| multiply | 3x | -1 |
|---|---|---|
| 4x | $12x^2$ | -4x |
| +5 | 15x | -5 |

$12x^2 - 4x + 15x - 5$

$12x^2 + 11x - 5$



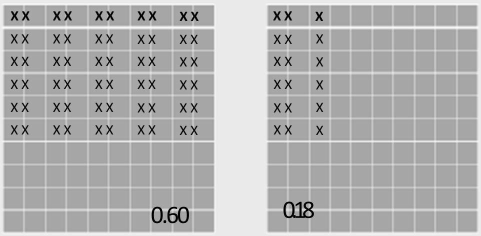
## Intervention Aspect of Place Value

- Language focused for conceptual building and to aid think alouds
- Physical models to show place value
  - http://ntnmath.com/video%20index/Index%20Videos/Polynomials/lesson%20108.htm
- Graphic Organizers for procedural memory and conceptual memory
- Possibility of a graduated sequence of instruction (CRA)



## Using Arrays for Decimals

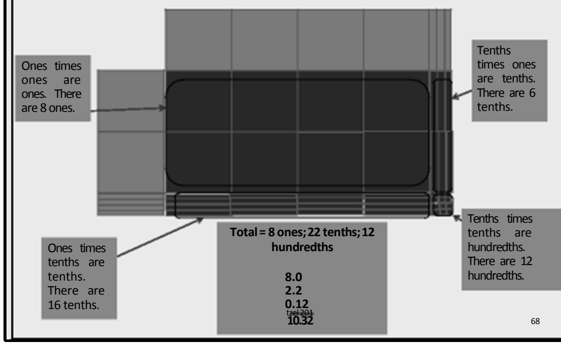
(1.3)(0.6)




Total = 0.78

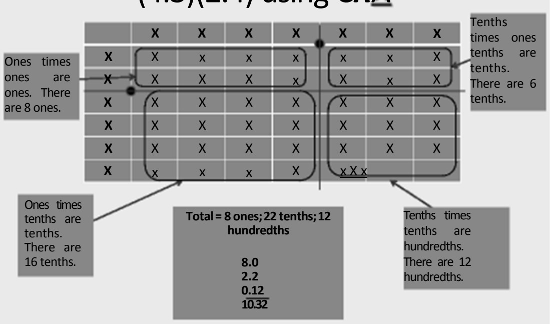


## (4.3)(2.4) using *C*RA






## (4.3)(2.4) using C*R*A






## (4.3)(2.4) using CR*A*

| multi | 4 | .3 |
|---|---|---|
| 2 | 8 | 0.6 |
| .4 | 1.6 | 0.12 |

Total = 8 ones; 22 tenths; 12 hundredths

8.0
2.2
0.12
10.32



## CRA delivery

(From Witzel, Riccomini, & Schneider, 2008)

- ***C***hoose the math topic to be taught;
- ***R***eview procedures to solve the problem;
  - ***A***djust the steps to eliminate notation or calculation tricks;
  - ***M***atch the abstract steps with an appropriate concrete manipulative;
  - ***A***rrange concrete and representational lessons;
  - ***T***each each concrete, representational, and abstract lesson to student mastery; and
  - ***H***elp students generalize what they learn through word problems.



## Find new ways to reach the students

“Whatever your difficulties in mathematics, I can assure you mine are far greater.”

**Albert Einstein**

Help build understanding of math early: Connect learning across time